Chapter 1

Ingenious Computation of Rational Numbers

The computation of rational numbers is similar to the computation of positive numbers. We should pay attention to the applications of operation properties (such as the commutative property of addition, the associative property of addition, the commutative property of multiplication, the associative property of multiplication, and the distributive property of multiplication) and skills in order to make the computation easy and convenient.

Example 1. Compute

$$48\frac{3}{5} - 18\frac{1}{4} - 1\frac{1}{3} + 0.25 + 3\frac{2}{3} - 2\frac{1}{3} - 30\frac{3}{5}.$$

Solution

$$\begin{aligned}
&48\frac{3}{5} - 18\frac{1}{4} - 1\frac{1}{3} + 0.25 + 3\frac{2}{3} - 2\frac{1}{3} - 30\frac{3}{5} \\
&= \left(-1\frac{1}{3} + 3\frac{2}{3} - 2\frac{1}{3}\right) + \left(48\frac{3}{5} - 30\frac{3}{5}\right) + \left(-18\frac{1}{4} + \frac{1}{4}\right) \\
&= \left(-\frac{1}{3} + \frac{2}{3} - \frac{1}{3}\right) + (48 - 30 - 18) + \left(\frac{3}{5} - \frac{3}{5}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) \\
&= 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

Remark: During addition and subtraction, we should pay attention to the application of the commutative and associative properties. We can change the order of operations in the computation, making the intermediate result an integer or even "cancel out." "Cancel out" means the sum of two opposite numbers is zero, such as the sum of $\frac{3}{5}$ and $-\frac{3}{5}$ and the sum of $-\frac{1}{4}$ and $\frac{1}{4}$ in this example. But during the computation, be careful of the signs and ensure that there are no mistakes. For example, $-\frac{1}{3}$ and $-\frac{1}{3}$ cannot cancel out, but their sum can, giving $\frac{2}{3}$.

Example 2. Compute

$$-2.5 \div 0.75 \times \left(-\frac{1}{5}\right) \times \left(-1\frac{3}{4}\right) \div (-1.4) \times \left(-\frac{3}{5}\right) \times \frac{2}{3}.$$

Solution

$$\begin{aligned} &-2.5 \div 0.75 \times \left(-\frac{1}{5}\right) \times \left(-1\frac{3}{4}\right) \div (-1.4) \times \left(-\frac{3}{5}\right) \times \frac{2}{3} \\ &= -\frac{5}{2} \div \frac{3}{4} \times \frac{1}{5} \times \frac{7}{4} \div \frac{14}{10} \times \frac{3}{5} \times \frac{2}{3} \\ &= -\frac{10}{3} \times \frac{1}{5} \times \frac{7}{4} \times \frac{10}{14} \times \frac{3}{5} \times \frac{2}{3} \\ &= -\frac{1}{3}. \end{aligned}$$

Remark: During multiplication and division, we should pay attention to the sign of the result: the multiplication of an odd number of negative numbers is a negative number; the multiplication of an even number of negative numbers is a positive number. Usually, we convert decimals to fractions, mixed fractions to improper fractions, and division to multiplication. Simplify the fraction first, and make the computation as easy as possible.

In addition, we should memorize some common results, such as $0.125 = \frac{1}{8}$, $0.375 = \frac{3}{8}$, and $0.75 = \frac{3}{4}$.

Example 3. Compute

$$\frac{1 \times 2 \times 3 + 2 \times 4 \times 6 + 7 \times 14 \times 21}{1 \times 3 \times 5 + 2 \times 6 \times 10 + 7 \times 21 \times 35}.$$

Solution

$$\begin{aligned} &\frac{1 \times 2 \times 3 + 2 \times 4 \times 6 + 7 \times 14 \times 21}{1 \times 3 \times 5 + 2 \times 6 \times 10 + 7 \times 21 \times 35} \\ &= \frac{1 \times 2 \times 3 \times (1 + 2 \times 2 \times 2 + 7 \times 7 \times 7)}{1 \times 3 \times 5 \times (1 + 2 \times 2 \times 2 + 7 \times 7 \times 7)} \\ &= \frac{2}{5}. \end{aligned}$$

Remark: During computation, we should pay attention to the application of the distributive property. If the numbers added have a common factor, we can first extract the common factor, then compute the parts with their common factors removed. In this example, the numerator has a common factor of $1 \times 2 \times 3$ and the denominator has a common factor of $1 \times 3 \times 5$; therefore, we can extract them and then simplify in order to make the computation easier. Moreover, if the common factor is a negative number, after extracting the common factor, the sign of each remaining part will change.

Example 4. Compute

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}.$$

Solution

$$\begin{aligned}
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\\
&\quad=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\left(\frac{1}{64}+\frac{1}{64}\right)-\frac{1}{64}\\
&\quad=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\left(\frac{1}{32}+\frac{1}{32}\right)-\frac{1}{64}\\
&\quad=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\left(\frac{1}{16}+\frac{1}{16}\right)-\frac{1}{64}\\
&\quad=\cdots\\
&\quad=\left(\frac{1}{2}+\frac{1}{2}\right)-\frac{1}{64}\\
&\quad=1-\frac{1}{64}\\
&\quad=\frac{63}{64}.
\end{aligned}$$

Remark: In this example, we can find a feature of the summation formula: every succeeding term is half of the preceding term. So, if we add the next term by itself, the result is simply the preceding term. Therefore, we ingeniously add $\frac{1}{64}$ to the formula and compute. Of course, since we add $\frac{1}{64}$ to the formula, we must also subtract $\frac{1}{64}$ at the end.

Example 5. Compute

(1) $1+2+3+4+\cdots+2007+2008$;
(2) $1-2+3-4+\cdots+2007-2008$.

Solution

(1) Let $S = 1+2+\cdots+2007+2008$. Then, $S = 2008+2007+\cdots+2+1$. After adding these two equations, we have

$$\begin{aligned}2S &= (1+2008)+(2+2007)+\cdots+(2007+2)+(2008+1)\\ &= \underbrace{2009+2009+\cdots+2009+2009}_{2008}\\ &= 2009\times 2008.\end{aligned}$$

Then, we get $S = \frac{2009\times 2008}{2} = 2017036$. To conclude, the original formula is equal to 2017036.

(2)

$$\begin{aligned}&1-2+3-4+\cdots+2007-2008\\ &= (1-2)+(3-4)+\cdots+(2007-2008)\\ &= \underbrace{(-1)+(-1)+\cdots+(-1)}_{1004}\\ &= -1004.\end{aligned}$$

Remark: The feature of problem (1) is that the difference between any two consecutive terms is the same value. We call such a sequence an arithmetic sequence. In other words, if a sequence $a_1, a_2, \ldots, a_n$ satisfies the condition that $a_{i+1} - a_i = d$ holds for all $i = 1, 2, \ldots, n-1$, then we call such a sequence an arithmetic sequence. Here, a_1 is called the first term, a_n is called the last term, and d is called the common difference. The formula for computing $a_1 + a_2 + \cdots + a_n$ is

$$\text{Sum} = \frac{(\text{the first term} + \text{the last term}) \times \text{the number of terms}}{2}.$$

Now, we can solve problem (1) using this formula.

Sometimes, the number of terms is not shown in the problem. We can compute it using this formula:

$$\text{the number of terms} = \frac{\text{the last term} - \text{the first term}}{\text{the common difference}} + 1.$$

For problem (2), we combine the two adjacent terms to simplify the computation. This is determined by using the feature of this problem. When solving problems, do not rush to the computation; rather, we should first observe the feature of the problem, and then start solving it from the feature, which can make half the effort, twice the result.

Example 6. Compute

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{1999\times 2000}.$$

Solution

$$\begin{aligned}
&\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{1999\times 2000}\\
&\quad=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{1999}-\frac{1}{2000}\right)\\
&\quad=1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{1999}+\frac{1}{1999}\right)-\frac{1}{2000}\\
&\quad=1-\frac{1}{2000}\\
&\quad=\frac{1999}{2000}.
\end{aligned}$$

Remark: During addition and subtraction, by using the feature of numbers, we can split every number into two, after which some of them can cancel out. We call this the "split method." In this example, we split $\frac{1}{n\times(n+1)}$ into $\frac{1}{n}-\frac{1}{n+1}$, which is

$$\frac{1}{n\times(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

There are some other ways of splitting:

(1) $\frac{d}{n\times(n+d)}=\frac{1}{n}-\frac{1}{n+d}$ or $\frac{1}{n\times(n+d)}=\frac{1}{d}(\frac{1}{n}-\frac{1}{n+d})$. Usually, this is applied when the factors of the denominators form an arithmetic sequence with common difference d.

(2) $\frac{1}{n\times(n+1)\times(n+2)}=\frac{1}{2}\times[\frac{1}{n\times(n+1)}-\frac{1}{(n+1)\times(n+2)}]$.

Example 7. Adding $\frac{1}{2}$ of 2002 to 2002, we get the first number. Then, adding $\frac{1}{3}$ of the first number to itself, we get the second number. Then, adding $\frac{1}{4}$ of the second number to itself, we get the third number, and so on in the same way until we add $\frac{1}{2002}$ of the current number to itself to get the last number. What is the last number?

Solution Since 2002 plus $\frac{1}{2}$ of it is $2002\times(1+\frac{1}{2})$, and then we add $\frac{1}{3}$ of this number to get $2002\times(1+\frac{1}{2})\times(1+\frac{1}{3})$, and so on, and the last number is equal to

$$\begin{aligned}
&2002 \times \left(1+\frac{1}{2}\right) \times \left(1+\frac{1}{3}\right) \times \cdots \times \left(1+\frac{1}{2002}\right)\\
&\quad = 2002 \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{2003}{2002}\\
&\quad = 2002 \times \frac{2003}{2}\\
&\quad = 2005003.
\end{aligned}$$

Remark: During multiplication, canceling the number that appears both in the numerator and denominator can greatly simplify the computation.

Reading

Rational Numbers

If you are new to rational numbers, then you might ask: Why are numbers of the form $\frac{m}{n}$ (where m and n are integers, $n \neq 0$) called rational numbers? Since there are rational numbers, are there irrational numbers?

Usually, it makes sense to give a thing a name. For example, the word "negative" of a negative number has the meaning of debt, and its meaning is exactly the opposite of the word "positive" of a positive number. The reason why a rational number is called so is unreasonable. It stems from a mistake in translation.

In the 19th century, when Western science was introduced to China, the Chinese mathematician Li Shanlan (1811–1882) translated rational function and irrational function into proportional and non-proportional formulas, respectively, when translating British mathematician De Morgan's *Algebra*. The understanding of these two names was completely correct, and the translated names were also correct because "proportional" refers to ratio. But more than 10 years later, when another mathematician, Hua Hengfang (1833–1902), translated Wallace's *Algebra*, he mistranslated rational and irrational as reasonable and unreasonable, respectively, which did not match the original meaning, but they became widely spread, even into Japan. Now, both China and Japan use these incorrect translations in education and academia.

As a rational number is a number that can be represented as $\frac{m}{n}$ (where m and n are integers, $n \neq 0$), a number that cannot be represented as $\frac{m}{n}$ is an irrational number. Such numbers exist. For example, the ancient Greeks discovered that the side length of a square is an irrational number if its area is 2.

Exercises

Compute:

1. $31\frac{2}{7} - 22\frac{6}{13} + 4\frac{5}{7} + 11\frac{6}{13}$.
2. $5\frac{6}{11} - 3.125 - 7\frac{4}{7} - 3\frac{4}{11} + 8\frac{1}{8} - 3\frac{6}{7} - 2\frac{2}{11} + 6\frac{3}{7}$.
3. $-\frac{7}{11} \div 2.5 \times (-0.75) \div (-1\frac{2}{5}) \div \frac{3}{11} \times (-\frac{8}{13})$.
4. $3.825 \times \frac{1}{4} - 1.825 + 0.25 \times 3.825 + 3.825 \times \frac{1}{2}$.
5. $-7.2 \times 0.125 + 0.375 \times 1.1 + 3.6 \times \frac{1}{2} - 3.5 \times 0.375$.
6. $\frac{1}{2-\frac{1}{3-\frac{1}{4-\frac{1}{5}}}}$.
7. $1\frac{1}{2} + 3\frac{1}{4} + 5\frac{1}{8} + 7\frac{1}{16} + 9\frac{1}{32}$.
8. $\frac{1}{1999} + \frac{2}{1999} + \frac{3}{1999} + \cdots + \frac{1998}{1999}$.
9. $(7 + 9 + 11 + \cdots + 101) - (5 + 7 + 9 + \cdots + 99)$.
10. $9 + 99 + 999 + 9999 + 99999 + 999999$.
11. $3^{2000} - 5 \times 3^{1999} + 6 \times 3^{1998}$.
12. $(-1)^{1998} + (-1)^{1999} + (-1)^{2000} + (-1)^{2001}$.
13. $\frac{1}{5\times 9} + \frac{1}{9\times 13} + \frac{1}{13\times 17} + \cdots + \frac{1}{101\times 105}$.
14. $2002\frac{1}{2} - 2001\frac{1}{3} + 2000\frac{1}{2} - 1999\frac{1}{3} + \cdots + 2\frac{1}{2} - 1\frac{1}{3}$.
15. $\frac{1\times 2\times 3+2\times 4\times 6+4\times 8\times 12+7\times 14\times 21}{1\times 3\times 5+2\times 6\times 10+4\times 12\times 20+7\times 21\times 35}$.
16. $1 + 2\frac{1}{6} + 3\frac{1}{12} + 4\frac{1}{20} + 5\frac{1}{30} + 6\frac{1}{42} + 7\frac{1}{56}$.
17. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+100}$.